Performance Bounds in $\mathcal{H}_\infty$ Optimal Control for Stable SISO Plants With Arbitrary Relative Degree

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***Abstract*—This note deals with performance bounds for the $\mathcal{H}_\infty$-optimal control of discrete-time LTI plants. The case studied corresponds to stable scalar plants with arbitrary relative degree but no finite non-minimum phase zero. By using Nehari's Theorem and a reformulation of the standard Youla Parameterization a closed-form expression for the characteristic polynomial of the associated eigenvalue problem is obtained. Also, we derive an analytic expression for the optimal $\mathcal{H}_\infty$ cost as a function of the plant relative degree.**

***Index Terms*—$\mathcal{H}$-infinity control, discrete-time systems, performance bounds.**

I. Introduction

Performance bounds and fundamental limitations provide the control system designer with a general idea of what can or cannot be achieved when dealing with a particular control problem. The study of fundamental limitations has its roots in the work of Bode [1] and has been an active research field ever since. The results of this area cover a wide variety of cases, including SISO [2] and MIMO [3] systems, continuous time [4] and discrete time [5] systems, frequency domain [6] and time domain [7] integrals, and so on, but the underlying fact for all cases is that the use of feedback comes with an inescapable (fundamental) set of *trade-offs* that the designer must consider when posing any controller synthesis method. On the other hand, performance bounds is a more recent research area, dealing with the search of the best achievable performance of a control system when considering a particular index measuring this performance and a particular class of admissible controllers. The case of $\mathcal{H}_2$ cheap control is well documented [8]–[10] and existing results include closed-form expressions for the best achievable performance [9], [11], [12]. The key link between the results of performance bounds and fundamental limitations is given by the terms in which the *trade-offs* and the expressions for the optimal achievable performance (or its bounds) are written. In both cases some dynamical features of the plant to be controlled, namely non-minimum phase (NMP) zeros, unstable poles and time delays (and their directions in the multivariable case) play a fundamental role, depicting the restrictions and the difficulties that the designer has to deal with.

Although the $\mathcal{H}_2$ norm provides deeper physical insight into the optimization problem, the $\mathcal{H}_\infty$ norm is used when robustness is a primary issue. In particular, the minimization of a sensitivity $\mathcal{H}_\infty$ norm cares for stability margins by emphasizing the reduction of sensitivity peaks. However, despite the existence of many approaches to the solution of $\mathcal{H}_\infty$ problems, the existing results mainly focus on general cases and on algorithms to numerically solve the design problems, with scarce results concerning achievable limits for a given sensitivity minimization problem. A few cases of $\mathcal{H}_\infty$ performance bounds have been studied in recent works [3], [13]. Most of the existing results for this setting provide bounds on the best achievable performance for continuous time systems [6], [7], [14], expressing them in terms of the NMP zeros and unstable poles of the plant. Unfortunately, those results do not give explicit closed-form expressions for the bounds although they clearly suggest a behavior similar to the 2-norm case, where the optimal achievable performance can be expressed as a simple function of the dynamical features of the plant [12]. In particular, the case with at most two NMP zero (unstable poles) and any number of unstable poles (NMP zeros) has been treated in [15] using an operator approach to give closed-form expressions for the best achievable infinity norm of the sensitivities for SISO continuous time systems. Tracking performance of discrete time systems in different norms has been studied and reported in various articles, including [16]–[19].

In this note we deal with the case of stable discrete time scalar systems with an arbitrary number of infinite NMP zeros and we derive closed-form $\mathcal{H}_\infty$ performance bounds for a one degree-of-freedom control loop with intqegral action. The derivation of performance bounds in the presence of finite NMP zeros and unstable poles is fairly more complex and difficult to incorporate in a note. Furthermore, in these cases approximations and numerical techniques are required.

The layout of the note is the following. Section II provides background material. In Section III the problem to be solved is defined. Section IV summarizes the standard numerical solution. In Sections V and VI the main results of this paper are derived. In Section VII an example is presented, and the main conclusions are drawn in Section VIII.

II. Definitions and Notations

This section introduces the basic notation and some definitions used in this note. For any complex number z, $\Re\{z\}$ denote its real component, $\angle\{z\}$ its argument and $\overline{z}$ its conjugate. A full zero Blaschke product is defined as

$$B_c(z) = \prod_{i=1}^{n_c} \frac{z - c_i}{1 - c_i z}, \quad c_i \in \mathbb{C}. \tag{1}$$

We consider a SISO one degree-of-freedom control loop where $G(z)$ is the plant to be controlled and $C(z)$ is a proper stabilizing controller. The space $\mathcal{RH}_\infty$ corresponds to the set of real rational, proper, and stable transfer functions. The performance index of the closed loop will be the infinity norm of the sensitivity function

$$S(z) = (1 + G(z)C(z))^{-1} \tag{2}$$

weighted by a function $W(z)$.

III. Problem Definition

In this section we review a standard $\mathcal{H}_\infty$ model matching problem and restrict it to the case of interest. Consider the functional [20]

$$J_{\mathcal{H}_\infty}(X(z)) \triangleq \|W(z)(T(z) - V(z)X(z))\|_\infty \tag{3}$$

where $W(z)$ is a weighting function, $T(z)$ and $V(z)$ are fixed transfer functions and $X(z)$ is a varying parameter in $\mathcal{RH}_\infty$. The model matching problem is to find $X_{opt}(z)$ which satisfies

$$X_{opt}(z) = \arg \min_{X(z)\in\mathcal{RH}_\infty} \|J_{\mathcal{H}_\infty}(X(z))\|_\infty \tag{4}$$

i.e., the parameter that achieves the minimum model-matching error $\gamma_{opt} \triangleq \min J_{\mathcal{H}_\infty}$. In this paper we work with LTI discrete time plants that are stable and have no finite NMP zeros. If we choose

$$W(z) = \frac{z}{z-1}, \quad T(z) = 1, \quad V(z) = z^{-\ell} \tag{5}$$

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where ℓ is the relative degree of $G(z)$, then (3) yields the infinity norm of the weighted sensitivity function. It can also be interpreted as the tracking error, in the frequency domain, of a one degree-of-freedom control loop when the reference is a unitary step. Using (5) in (3) our cost function becomes

$$J = \left\| \frac{z}{z-1} \left(1 - \frac{X(z)}{z^\ell} \right) \right\|_\infty \tag{6}$$

since the sensitivity of the loop can be parameterized as

$$S(z) = 1 - \frac{X(z)}{z^\ell}. \tag{7}$$

This expression is a particular case of the general parameterization [21]

$$S(z) = S_o(z) - E_c(z)^{-1} E_p(z)^{-1} X(z) \tag{8}$$

where $E_c(z)$ and $E_p(z)$ are the zero and pole plant interactors, respectively. For stable plants with relative degree ℓ and no finite NMP zeros

$$E_c(z) = z^\ell, \quad E_p(z) = 1 \tag{9}$$

and $S_o(z)$ is the loop sensitivity achieved with any stabilizing proper controller. Given that the plant is stable, we have chosen this controller to be $C_o(z) = 0$, which yields $S_o(z) = 1$.

The control loop has zero tracking error for constant references and is guaranteed to be admissible (that is, internally stable with a proper controller) if and only if the parameter $X(z)$ is written as [21]

$$X(z) = 1 + \frac{z-1}{z} \tilde{X}(z) \tag{10}$$

where $\tilde{X}(z) \in \mathcal{RH}_\infty$. This requirement ensures that the functional J is well defined. If the identity $\|z^\ell H(z)\|_\infty = \|H(z)\|_\infty$ is used, then (6) can be written as

$$J = \left\| \frac{z}{z-1}(z^\ell - 1) - \tilde{X}(z) \right\|_\infty. \tag{11}$$

The functional given in (11) will be considered as the model matching problem to be solved.

IV. Solution of the Model Matching Problem

Next we briefly review the standard solution to the model matching problem based on the well known Nehari theorem [20].

Assume that $R(z)$ is a transfer function that satisfies

$$R(z) = R_1(z) + R_2(z) \tag{12}$$

with $R_1(z)$ analytic in $|z| \leq 1$ and $R_2(z) \in \mathcal{RH}_\infty$. Let a minimal state-space realization of $R_1(1/z)$ be

$$R_1(1/z) = [\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, D] \tag{13}$$

and define the controllability and observability gramians of $R_1(1/z)$, $\boldsymbol{L_c}$ and $\boldsymbol{L_o}$, respectively, as the solutions of the discrete time Lyapunov equations

$$\boldsymbol{L_c} - \boldsymbol{A L_c A^T} = \boldsymbol{B B^T}, \quad \boldsymbol{L_o} - \boldsymbol{A^T L_o A} = \boldsymbol{C^T C}. \tag{14}$$

Lemma 1: The model matching problem

$$\min_{\tilde{X}(z) \in \mathcal{RH}_\infty} \left\| R(z) - \tilde{X}(z) \right\|_\infty \tag{15}$$

has an unique solution given by

$$\tilde{X}_{opt}(z) = R(z) - \gamma_{opt} \Theta(z) \tag{16}$$

where γ_{opt} is the square root of the largest eigenvalue λ of the product $\boldsymbol{L_c L_o}$, and $\Theta(z)$ is an all-pass transfer function with unity dc-gain given by

$$\Theta(z) = B_c^R(z) \frac{w_0 z^{k-1} + w_1 z^{k-1} + \ldots + w_{k-1}}{w_0 + w_1 z + \ldots + w_{k-1} z^{k-1}} \tag{17}$$

with k equal to the number of poles of $R_1(1/z)$, $\boldsymbol{w} = [w_0 \ w_1, \ldots, w_{k-1}]^T$ being the eigenvector associated to λ and $B_c^R(z)$ is a Blaschke product with zeros located at the poles of $R_1(1/z)$

Proof: This is a straightforward extension to the discrete time case of the results in [20]. ■

Lemma 1 gives the structure of the optimal parameter for a particular model matching problem and a procedure to compute it. It is clear that the complexity of the associated eigenvalue problem grows with k. Other approaches to solve the problem, like Nevanlinna-Pick interpolation [22], suffer the same complexity issue. This has been one of the motivations for the researchers in the field to tackle the problem through a numerical and algorithmic approach.

V. Performance Bounds

As seen on the previous section, there is a unique solution to the model matching problem when it is posed as in (15). The problem that we are interested in is the minimization of the cost function given in equation (11), and it translates into the form of (15) when

$$R(z) = \frac{z}{z-1}(z^\ell - 1) = \sum_{i=1}^{\ell} z^i. \tag{18}$$

In this case $R_1(z) = R(z)$, and a minimal realization (observable form) for $R_1(1/z)$ is

$$\boldsymbol{A} = \begin{bmatrix} 0 & & & \boldsymbol{0} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \boldsymbol{0} & & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$
$$\boldsymbol{C} = [0 \quad \ldots \quad 0 \quad 1], \quad D = 0 \tag{19}$$

where $\boldsymbol{A} \in \mathbb{R}^{\ell \times \ell}$, $\boldsymbol{B} \in \mathbb{R}^{\ell \times 1}$ and $\boldsymbol{C} \in \mathbb{R}^{1 \times \ell}$. The gramians for this representation are given by

$$\boldsymbol{L_c} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & \ell \end{bmatrix}, \quad \boldsymbol{L_o} = \boldsymbol{I}_{\ell \times \ell} \tag{20}$$

and their product satisfies

$$\boldsymbol{P L_c L_o} = \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ \boldsymbol{0} & & 1 \end{bmatrix}, \quad \text{for } \boldsymbol{P} = \begin{bmatrix} 1 & & & \boldsymbol{0} \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ \boldsymbol{0} & & -1 & 1 \end{bmatrix}. \tag{21}$$

Note that, the assumption of the plant having no unstable poles and no finite NMP zeros is crucial for the structures of the matrices in (20) and (21). The following result gives an expression for the characteristic polynomial of $\boldsymbol{L_c L_o}$, for any given value of ℓ.

Lemma 2: The characteristic polynomial for the matrix $\boldsymbol{L_c L_o}$, given in (20), satisfies the following recursive relation

$$p_{\ell+2}(x) = (2x-1)p_{\ell+1}(x) - x^2 p_\ell(x), \quad \ell \geq 0 \tag{22}$$

with $p_0(x) = 1$, and $p_1(x) = x - 1$.

Proof: See Appendix I. ■

Exact expressions for $p_\ell(x)$ and its largest root are given in the following lemma.

Lemma 3: The characteristic polynomial $p_\ell(x)$ for $\boldsymbol{L_c L_o}$, is given by

$$p_\ell(x) = \frac{1}{2}\left[\frac{\sqrt{4x-1}+j}{\sqrt{4x-1}}\left(\frac{2x-1+j\sqrt{4x-1}}{2}\right)^\ell + \frac{\sqrt{4x-1}-j}{\sqrt{4x-1}}\left(\frac{2x-1-j\sqrt{4x-1}}{2}\right)^\ell\right]. \quad (23)$$

Moreover, the largest root of $p_\ell(x)$ and the optimal achievable performance are

$$\lambda_\ell = \frac{1}{4}\sec^2\left(\frac{\ell\pi}{2\ell+1}\right), \quad \gamma_{opt} = \frac{1}{2}\sec\left(\frac{\ell\pi}{2\ell+1}\right). \quad (24)$$

Proof: See Appendix II. ■

VI. Closed Loop Dynamics

We can now study the resulting closed loop dynamics when using the optimal parameter (16) defined in Lemma 1. If we replace this expression in the sensitivity of the closed loop, given by (7) and (10), we have that

$$\begin{aligned} S_{opt}^{\mathcal{H}_\infty}(z) &= 1 - \frac{1}{z^\ell}\left(1 + \frac{z-1}{z}\tilde{X}_{opt}(z)\right) \\ &= \frac{z-1}{z}\gamma_{opt}\frac{B_c^R(z)}{z^\ell}\frac{w_0 z^{k-1} + w_1 z^{k-1} + \ldots + w_{k-1}}{w_0 + w_1 z + \ldots + w_{k-1}z^{k-1}}. \end{aligned} \quad (25)$$

In our case, $R_1(z)$ is improper and $R_1(1/z)$ has ℓ poles at the origin. This implies that $k = \ell$, $B_c^R(z) = z^\ell$ and consequently the closed loop polynomial is

$$\begin{aligned} A_{cl}(z) &= z\boldsymbol{\eta_\ell}(z)^T\boldsymbol{w}, \\ \boldsymbol{\eta_\ell}(z) &= [1 \quad z \quad \cdots \quad z^{\ell-1}]^T \end{aligned} \quad (26)$$

and $\boldsymbol{w}$ is the eigenvector obtained by solving

$$\boldsymbol{L_c L_o w} = \lambda_\ell \boldsymbol{w}. \quad (27)$$

We have that, multiplying from the left by $\boldsymbol{P}$ defined in (21), we obtain

$$\begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ \mathbf{0} & & 1 \end{bmatrix}\boldsymbol{w} = \lambda_\ell \begin{bmatrix} 1 & & & \mathbf{0} \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ \mathbf{0} & & -1 & 1 \end{bmatrix}\boldsymbol{w}. \quad (28)$$

Since the system (28) is singular, we can choose any nonzero $w_{\ell-1}$, and then

$$w_{\ell-j} = w_{\ell-j+1} - \left(\sum_{i=\ell-j+1}^{\ell-1} w_i\right)/\lambda_\ell \quad (29)$$

for $j = 2, 3, \ldots, \ell$. These are analytic expressions for the coefficients of the closed loop polynomial in terms of λ_ℓ. The next result gives a more compact expression for these coefficients.

Lemma 4: For $\ell > 1$, the eigenvector $\boldsymbol{w}$ defined by the eigenvalue problem (27) can be computed as

$$w_{\ell-j} = \frac{p_{j-1}(\lambda_\ell)}{\lambda_\ell^{j-1}} \quad (30)$$

where the polynomials $p_i(x)$ are given in (23), and $j = 1, \ldots, \ell$.

Proof: By fixing $w_{\ell-1} = 1$ and a direct use of Cramer's Rule in (28). ■

The controller achieving this performance can be computed from (25) and (2) as

$$C_{opt}(z) = G(z)^{-1}\left(S_{opt}^{\mathcal{H}_\infty}(z)^{-1} - 1\right). \quad (31)$$

VII. Example

We consider the case $\ell = 7$, i.e. any stable plant with no finite NMP zeros and relative degree equal to 7. According to lemma 3, the optimal performance is given by (24)

$$\gamma_{opt} = \sqrt{\lambda_7} = \frac{1}{2}\sec\left(\frac{7\pi}{15}\right) = 4.7834 \quad (32)$$

and the associated eigenvector results

$$\boldsymbol{w} = [0.2091 \quad 0.4090 \quad 0.5910 \quad 0.7472 \quad 0.8707 \quad 0.9563 \quad 1]^T. \quad (33)$$

Now, the optimal sensitivity achieved is

$$S_{opt}^{\mathcal{H}_\infty}(z) = \frac{z-1}{z}\frac{\boldsymbol{\nu_7}(z)^T\boldsymbol{w}}{\boldsymbol{\eta_7}(z)^T\boldsymbol{w}}, \quad \boldsymbol{\nu_7}(z) = z^6\boldsymbol{\eta_7}(z^{-1}). \quad (34)$$

VIII. Conclusion

In this note we have derived a closed-form expression for an optimal achievable performance based on an infinity norm index. The optimization yields an admissible controller for a stable plant with arbitrary relative degree and no NMP finite zeros, yielding zero error for constant references. An analytic expression has been also derived for the closed loop characteristic polynomial. The main result shows that the optimal achievable performance is a function of the relative degree of the plant. Furthermore, the results in this case show that the stable and MP part of the plant plays no role when computing the ultimate limit of performance.

Appendix I
Proof of Lemma 2

First, we prove that $p_\ell(x)$ satisfies

$$\begin{aligned} p_\ell(x) = (x-1)p_{\ell-1}(x) - xp_{\ell-2}(x) - x^2 p_{\ell-3}(x) \\ - \ldots - x^{\ell-3}p_2(x) + x^{\ell-2}(1-2x) \end{aligned} \quad (35)$$

for $\ell \geq 4$. For $\ell = 2$, we have

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow \boldsymbol{PL_c L_o} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (36)$$

Now, since $\boldsymbol{P}$ is non-singular, the eigenvalue problems

$$\det(x\boldsymbol{I} - \boldsymbol{L_c L_o}) = 0, \quad \det(x\boldsymbol{P} - \boldsymbol{PL_c L_o}) = 0 \quad (37)$$

are equivalent, and the characteristic polynomial is computed as

$$\begin{aligned} p_2(x) &= \left|\begin{bmatrix} x & 0 \\ -x & x \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right| \\ &= \begin{vmatrix} x-1 & -1 \\ -x & x-1 \end{vmatrix} = x^2 - 3x + 1. \end{aligned} \quad (38)$$

Analogously, for $\ell = 3$

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \boldsymbol{PL_c L_o} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (39)$$

The characteristic polynomial is now given by

$$\begin{aligned} p_3(x) &= \left| \begin{bmatrix} x & 0 & 0 \\ -x & x & 0 \\ 0 & -x & x \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| \\ &= \begin{vmatrix} x-1 & -1 & -1 \\ -x & x-1 & -1 \\ 0 & -x & x-1 \end{vmatrix} \\ &= (x-1)p_2(x) + x(1-2x) \\ &= x^3 - 6x^2 + 5x - 1. \end{aligned} \tag{40}$$

For $p_\ell(x)$, when $\ell \geq 4$, we have (21), and the characteristic polynomial is

$$\begin{aligned} p_\ell(x) &= \left| \begin{bmatrix} x & & & \mathbf{0} \\ -x & \ddots & & \\ & \ddots & \ddots & \\ \mathbf{0} & & -x & x \end{bmatrix} - \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ \mathbf{0} & & 1 \end{bmatrix} \right| \\ &= \begin{vmatrix} x-1 & -1 & \cdots & -1 \\ -x & \ddots & \ddots & \vdots \\ & \ddots & \ddots & -1 \\ \mathbf{0} & & -x & x-1 \end{vmatrix} \\ &= (x-1)p_{\ell-1}(x) + x\det \boldsymbol{M_{\ell-1}} \end{aligned} \tag{41}$$

where $\boldsymbol{M_{\ell-1}} \in \mathbb{R}^{\ell-1\times\ell-1}$ is given by

$$\boldsymbol{M_{\ell-1}} = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -x & x-1 & \ddots & \vdots \\ & \ddots & \ddots & -1 \\ \mathbf{0} & & -x & x-1 \end{bmatrix}. \tag{42}$$

It is clear that $\det \boldsymbol{M_{\ell-1}} = -p_{\ell-2} + x \det \boldsymbol{M_{\ell-2}}$. Furthermore, since $\det \boldsymbol{M_2} = 1 - 2x$, we have that the last line of (41) yields (35). Finally, $p_{\ell+2}(x) - xp_{\ell+1}(x)$ yields (22) with the corresponding initial conditions. ■

APPENDIX II
PROOF OF LEMMA 3

The expression for $p_\ell(x)$ given in (23) is derived as the solution of the second order difference equation defined in (22). If we denote $P_x(z)$ as the z-transform of $p_\ell(x)$, (22) yields

$$z^2P_x(z) - z^2p_0 - zp_1 = (2x-1)\left(zP_x(z) - zp_0\right) - x^2P_x(z). \tag{43}$$

Substituting the initial conditions $p_0 = 1$ and $p_1 = x - 1$, $P_x(z)$ is given by

$$\begin{aligned} P_x(z) &= \frac{z(z-x)}{z^2 - (2x-1)z + x^2} \\ &= z\left(\frac{k_1}{z - r_1} + \frac{k_2}{z - r_2}\right) \end{aligned} \tag{44}$$

where

$$\begin{aligned} r_{1,2} &= \frac{1}{2}(2x - 1 \pm j\sqrt{4x-1}), \quad \text{and} \\ k_{1,2} &= \frac{1}{2} \pm \frac{j}{2\sqrt{4x-1}}. \end{aligned} \tag{45}$$

Now, if we recall that the inverse z-transform of (44) is $p_\ell = k_1(r_1)^\ell + k_2(r_2)^\ell$, then we have (23). Since $r_1 = \overline{r_2} = r$ and $k_1 = \overline{k_2} = k$, the roots of $p_\ell(x)$ satisfy

$$\Re\left\{k(r)^\ell\right\} = 0, \quad \cos\left(\angle\left\{k(r)^\ell\right\}\right) = 0. \tag{46}$$

From (23), we have that

$$\angle\left\{k(r)^\ell\right\} = \arctan\frac{1}{\sqrt{4x-1}} + \ell\arctan\frac{\sqrt{4x-1}}{2x-1}. \tag{47}$$

Let $u = \sqrt{4x-1}$, then $u^2/2 - 1/2 = 2x - 1$, thus

$$\begin{aligned} \angle\left\{k(r)^\ell\right\} &= \arctan\frac{1}{u} + \ell\arctan\frac{2u}{u^2-1} \\ &= (2\ell+1)\arctan\frac{1}{u} \\ &= (2\ell+1)\left(\frac{\pi}{2} - \arctan u\right). \end{aligned} \tag{48}$$

Then, (24) results from the fact that the largest root of the polynomial satisfies

$$(2\ell+1)\left(\frac{\pi}{2} - \arctan\sqrt{4\lambda_\ell - 1}\right) = \frac{\pi}{2}. \tag{49}$$

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